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A family of groups generalising the Poincaré group and their physical applications

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Abstract. A family of groups $P^{(m,n)}$ parametrised by non-negative integers $m > n$ is studied generalising the Poincaré group $P^{(1,1)}$. The action of finite-dimensional irreducible real representations of SL(2, C) is used to form semi-direct products $R^N \otimes SL(2, C)$. We define the complete list of unitary irreducible representations for each $P^{(m,n)}$ by finding all subgroups of SL(2, C) which are little groups; some of the subgroups do not occur in the Poincaré group. The geometry of the SL(2, C) action and the classification of SL(2, C) invariant tensors is considered. These groups are appropriate symmetry groups for field theories in higher dimensions and generalise the notion of elementary relativistic quantum systems.

1. Introduction

We shall be guided by the idea that irreducible unitary representations of a physical transformation group can be interpreted as elementary quantum systems. In particular, since Wigner's classic work (Wigner 1930), relativistic elementary particles are defined by the unitary irreducible representations of the Poincaré group in which the Lorentz group, L, acts on the four-dimensional momentum space via its irreducible vector representation $R^{(1,1)} = \Lambda$. The structure of an elementary system is understood in terms of the inducing structure of the associated representation. The type of particle is determined by the particular subgroup of L, which serves as the little group for induction, since the internal quantum numbers are defined by the representations of that little group. It is well known that only three non-trivial subgroups of the Lorentz group occur as little groups: SO(3), SO(2, 1) and E_2. The corresponding relativistic elementary systems are massive particles with the spin internal degrees of freedom, massless particle with helicity or 'continuous spin', and imaginary mass particles with an internal quantum number having infinitely many values.

The first two are identified with real elementary particles, while the last one may be used in describing the virtual exchange of a negative energy momentum squared. But there is *more structure* in the Lorentz group not accounted for in this picture. This suggests that perhaps another group containing the Lorentz group, but with more parameters than the Poincaré group, would describe a larger set of elementary systems. A simple way to generalise from the Poincaré group is to keep the Lorentz group, but allow it to act irreducibly on vector spaces of higher dimension than four. Physically this may be interpreted as attributing to 'space–time' more dimensions than four, the four dimensions we perceive being a collapse or projection of these.

We make more remarks about the physical significance of these generalisations at the end of this section and at the end of the paper.

The square of the four-momentum distinguishes among the three types of particles defined by the Poincaré group. Its value is determined by the unique generating invariant tensor under the $R^{(1,1)}$ action of L, namely the metric tensor. By contrast, in our generalised groups we find a variety of generating sets of invariant tensors. If one considers the metric as a polynomial, p, on the four-dimensional Minkowski space then the dimension of the level surfaces, the set of four-vectors, u, such that $p(u)=k$, for some constant k, is three. The dimension of the various non-trivial orbits is also three. In general the dimension of an orbit under any Lorentz group action is six, the number of parameters of the group, minus the number of parameters of the little group associated with that orbit. All the non-trivial orbits under L in the Poincaré group have dimension $3=6-3$ since all the non-trivial subgroups have three parameters. Because of this, the single invariant polynomial defined by the metric tensor can classify the orbits with its level surfaces, since they foliate the four-dimensional space to three-dimensional surfaces. In contrast with this, in our generalised groups we have found a number of new subgroups occuring as little groups which have one, two and three parameters implying orbits of five, four, and three dimensions all in the same space.

We generalise the Poincaré group by using other irreducible finite-dimensional Lorentz group actions $R^{(m,n)}$ than the four-dimensional defining one, $R^{(1,1)}$. We denote the generalised group having the action $R^{(m,n)}$ by $\mathrm{P}^{(m,n)}$. So $\mathrm{P}^{(1,1)}$ is the Poincaré group. We view the abstract Lorenz group as the physical relativistic transformation group for 'space–time'. We consider 'space–times' of arbitrary finite dimensions on which the Lorentz group acts irreducibly. (The physical meaning of these various dimensions we leave undetermined, hoping to find it through the group action.) We approach the study of the groups $\mathrm{P}^{(m,n)}$ with the desire to encompass more generalised quantum systems. We find among the irreducible unitary representations of this family of groups some subgroups of the Lorentz group not occurring as little groups for representations of the Poincaré group.

The dimension of the vector space in $\mathrm{P}^{(m,n)}$ is $N=(m+1)(n+1)$. We can use the group to give physical meaning to a given vector either by finding its little group, thus using an algebraic property, or by studying the orbit it lies in under the group action and the values of a generating set of invariant polynomials on that orbit, a geometric property. The algebraic correspondence between a direction in a carrier space and the subgroup of L which fixes a vector in that direction we classify completely. These results alone we find are inadequate for determining the physical significance of dimensions in our generalised 'space–times'. We initiate the study of orbit geometries by beginning a classification of invariant polynomials.

As we shall see, the groups which are the most direct generalisations of the Poincaré group have non-trivial orbits of dimension three, four and five all in the same space. This contrasts sharply with the Poincaré group where all non-trivial orbits are three dimensional. We know that an invariant polynomial must be constant on any orbit. Thus an orbit must be contained in a level surface of every invariant polynomial, i.e. the surface defined as the set of all points where that polynomial has a fixed value. We do not know the extent to which an orbit can be determined by intersecting level surfaces of invariant polynomials. This in itself is an interesting apparently unsolved problem related to Hilbert's fourteenth problem: is the ring of polynomials which are invariant under a given matrix group finitely generated? in our case there will be a finite set generating all polynomials under our matrix group defined by $R^{(m,n)}$ (Fogarty 1969). We do not know whether an orbit under a given group can be defined by intersecting level surfaces of invariant polynomials chosen from a generating set. The fact that there are a variety of dimensions for the orbits of the group suggests a very complex, rich structure.

Our generalised groups relate to other work in various ways. In the theories of asymptotically flat spaces the Bondi–Metzner–Sachs group (McCarthy 1972, 1973, 1975, 1978, Piard 1977) has been found useful in which there are an infinite number of supermomenta and the Lorentz group acts irreducibly on these. Our groups $\mathrm{P}^{(m,n)}$ lie between the extremes of the Poincaré group and the BMS group. A simple reducible case where SL(2, C) acts on the relativisitic phase-spaces of two four-vectors x_μ and p_μ via the direct product $\Lambda\times\Lambda$ has been investigated by Piron (1979). We are interested in more general phase spaces, namely the carrier spaces of the various irreducible $R^{(m,n)}$. Among other applications might be an interacting particle theory in which the momenta of two or more particles are transformed together under a irreducible representation $R^{(m,n)}$. Recently there is renewed interest in field theories in higher-dimensional spaces following the ideas of the Kaluza–Klein theory in five dimensions (Salam and Strathdee 1982, Cremmer and Julia 1979). These theories have higher-dimensional translations, and when reduced to the Minkowski space they give rise to additional particles or degrees of freedom, just as in our case, as we will see, we get additional four-vectors when the little groups are considered acting in Minkowski space.

2. Definition of the generalised Poincaré groups

The general analytic finite-dimensional irreducible complex representation $R^{(m,n)}$ of SL(2, C) can be realised as acting in the $(m+1)(n+1)$ dimensional space of polynomials, $u(z,\bar z)$, in the variables z and $\bar z$ of degree less than or equal to m in z, and less than or equal to n in $\bar z$ as follows (see e.g. Barut and Raçzka 1977):

$$T_g^{(m,n)}u(z,z)=(\beta z+\delta)^m(\overline{\beta z+\delta})^n u\left(\frac{\alpha z+\gamma}{\beta z+\delta},\overline{\frac{\alpha z+\gamma}{\beta z+\delta}}\right)\qquad g=\begin{pmatrix}\alpha\beta\\ \gamma\delta\end{pmatrix}\in\mathrm{SL}(2,C).$$

In generalising the Poincaré group we want SL(2, C) to act irreducibly on a real vector space. We find that all smooth real representations, $R^{(m,n)}$, are built out of the above complex representations via:

Theorem. The real analytic irreducible finite-dimensional representations, $R^{(m,n)}$ of SL(2, C) are all either equivalent to $R^{(m,m)}$ for some m, or to $T^{(m,n)}\oplus T^{(n,m)}$ for some m and n, $m\neq n$. In particular, a representation $T^{(m,n)}\oplus T^{(n,m)}$, $m\neq n$, of SL(2, C) is equivalent to the representation obtained from making $T^{(m,n)}$ real by doubling the dimension.

To prove this we first show that if $m\neq n$, $T(m,n)$ cannot be equivalent to a real representation. This is done by showing there always must be some matrix element with an imaginary component. Next we show that if $T^{(m,n)}\oplus D$ is equivalent to a real representation then $T^{(n,m)}\subset D$, and that $T^{(m,n)}\oplus T^{(n,m)}$ is real irreducible and equivalent to making $T^{(m,n)}$ real by doubling the dimensions. The latter is done by defining an automorphism of the Lie algebra which makes the action on the vector space real. Finally we define a similar automorphism on the Lie algebra to $T^{(m,m)}$ (see also Weiss 1981).

Using this theorem we know explicitly the matrix elements of the general real N-dimensional representation. We define our generalised Poincaré group $\mathrm{P}^{(m,n)}$ as the semi-direct product of SL(2, C) with R^N via the action $R^{(m,n)}$ defined by

$$R^{(m,n)} = \begin{cases} T^{(m,n)} \otimes T^{(n,m)}, & \text{if } m \neq n \\ T^{(m,m)}, & \text{if } m = n. \end{cases}$$

The group action then is

$$a \in R, \qquad g \in \mathrm{SL}(2, C) \to (a, g)(a', g') = (a + R_g^{(m,n)}(a'), gg').$$

We have a different group $P^{(m,n)}$ for all $m \geqslant n$.

3. Little groups in the representations of $\mathrm{P}^{(m,n)}$

We determine the representations of our generalised groups by inducing them from the little groups. In the case of the Poincaré group one has a full understanding of the orbits of the group action in R^4, the dual to Minkowski space. This allows one to choose standard vectors, find the little groups of those vectors, and induce representations. Unfortunately it is extremely difficult to analyse the group action in R^N for an arbitrary N. In our approach we determine first which subgroups of SL(2, C) fix at least one vector in some representation $R^{(m,n)}$. Since all the connected subgroups of SL(2, C) have been classified (Shaw 1970), we can identify these by taking a particular parametrisation of a given subgroup and computing the form a vector must have to remain fixed under the action of that subgroup. This method is due to McCarthy who developed it for the BMS group (McCarthy 1972, 1973, 1975, 1978, Piard 1977). Then we can choose standard vectors and find their little groups. Subgroups which have only the trivial connected component can be analysed by assuming that at least one element is in Jordan normal form, and computing the possible fixed vectors of the subgroup generated by that one element. The results are summarised in table 1, showing the list of standard vectors and their little groups. The connected component of the identity in a little group is labelled as $K(Mj)$, $K(Nj)$ or $K(Ej)$, meaning it is characterised in Minkowski space as the group which *preserves* a j-dimensional subspace which is either Minkowski, null or Euclidean respectively. If the subgroup is defined as *fixing* a subspace rather than merely preserving it, the subgroup is labelled with a dot: $K(\dot{M}j)$, $K(\dot{N}j)$, $K(\dot{E}j)$. A j-dimensional Minkowski or null space is generated by a time-like or null vector, respectively, and $(j-1)$ orthogonal space-like vectors. A j-dimensional Euclidean space, of course, is generated by j orthogonal space-like vectors.

We see from table 1 that all little groups except SU(2), E_2, SU(1, 1) and SL(2, C) are new compared with the case of the Poincaré group. Only these four little groups are characterised as fixing a single vector in the Minkowski space. These groups only occur in $\mathrm{P}^{(m,n)}$ for $m = n$, namely the groups for which the complex representation $T^{(m,n)}$ is already equivalent to a real representation. In these cases, the dimension of $\mathrm{P}^{(m,n)}$ is an arbitrary square $(n+1)$ (Fogarty 1969). We see that the Poincaré group has the smallest dimension, $4=2^2$, of any group having these little groups, except the trivial case $P^{(0,0)}$ with dimension one.

There are four other connected subgroups which are connected little groups of at least one vector in R^N under some $R^{(m,n)}$ but are not the full connected component of a little group for any vector in the Minkowski space. These new little groups are characterised in the Minkowski space as either fixing or preserving a j-dimensional subspace. Thus, it is remarkable that these new little groups in R^N lead in the Minkowski space to a generalisation of the concept of stability group.

4. The representations of $P^{(m,n)}$

The next step is to determine the unitary irreducible representations of the little groups, and the resultant 'quantum numbers' generalising the internal quantum numbers spin or helicity. The group $P^{(m,n)}$ are algebraic groups and hence due to a theorem of Dixmier (1959) are regular. Thus Mackey's theorem implies that a Borel section of the orbits exists and every irreducible representation is induced. The representations of the little groups are all either well known or easily found by inducing from subgroups of the little groups themselves, but with two noteworthy exceptions: (4*b*) and (6*b*). Here the connected component of the group is adjoined with the matrix

$$g=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

generating a group with two connected components. In a unitary representation of one of these little groups it turns out that the operator J representing the group element g intertwines an irreducible representation of the connected component of the identity with its contragradient, thus acting as a metric. Therefore an irreducible representation of the group (4*b*) or (6*b*) is a direct sum of an irreducible and its contragradient of (4*a*) or (6*a*), respectively. A complete derivation of the unitary representations of the little groups is given elsewhere (Weiss 1981).

The complete characterisation of the induced representations requires some knowledge about the orbits of the little group in R^N. The orbits are $(6-l)$-dimensional manifolds, where l is the dimension of the little group. It is not clear whether they can be characterised by the invariant tensors of $R^{(m,n)}$ generalising the concept of 'mass-shell' for the Poincaré group. To do so, there would need to be at least N-$(6-l)$ of them. The geometry of these orbits and the theory of invariant tensors of $R^{(m,n)}$ is a new, complex and interesting study which we begin in this work. It turns out to be sufficient to find the invariant tensors of the complex representations $T^{(m,n)}$. This is obvious if $m=n$ for then $R^{(m,m)}\cong T^{(m,m)}$. If $m\neq n$, $R^{(m,n)}$ is formed from $T^{(m,n)}$ by separating real and imaginary parts. Now by an invariant tensor we mean a tensor product of generators of the vector group R^N which remains invariant under the SL(2, C) action. This is equivalent to commuting with the generators of SL(2, C). If a tensor commutes with the generator of a given one-parameter group $R_g^{(m,n)}$ it must commute with the real and imaginary parts of the generator of $T_g^{(m,n)}$. Conversely for a tensor to commute with $T_g^{(m,n)}$ it must commute with the real and imaginary parts separately. It is important to know if a tensor has a symmetric part for we want to use them to characterise the orbits. A totally antisymmetric tensor defines a trivial polynomial on R. We have:

Table 1. Little groups and their fixed vectors. The parameters have values γ, θ, a, $b \in R$; α, β, γ, $\delta \in C$; $k \in Z$.

Connected component of the identity	Form of elements of the full little group	$T^{m,n}$ representation where occurs	Fixed vector
(1) $K(\dot{N}_1)$	$\begin{pmatrix} e^{i\theta/2} & e^{i\theta/2}(a-ib)/\sqrt{2} \\ 0 & e^{-i\theta/2} \end{pmatrix}$	$m=n$	$z^m(\bar{z})^m$
(2) $K(\dot{N}_1, N_2)$	$\begin{pmatrix} \exp\left(i\dfrac{2\pi k}{m-n}\right) & \exp\left(i\dfrac{2\pi k}{m-n}\right)\dfrac{a-ib}{\sqrt{2}} \\ 0 & \exp\left[-i\dfrac{2\pi k}{m-n}\right] \end{pmatrix}$	$m \neq n$	$z^m(\bar{z})^n$
(3) $K(\dot{N}_2)$	(*a*) $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$	$m+n$ odd	$\left\{\begin{array}{l} {}^{\dagger}\sum_t A_t u_t \\ \max\{m,n\} \leqslant t \leqslant m+n \\ \text{where:} \\ u_t = (z = \bar{z})^{m+n-t} z^{t-n} (\bar{z})^{t-m} \end{array}\right.$
	(*b*) $\begin{pmatrix} i^k & i^k a \\ 0 & i^{-k} \end{pmatrix}$	$(n-m)/2-t$ even	
	(*c*) $\begin{pmatrix} (-1)^k & a \\ 0 & (-1)^k \end{pmatrix}$	$n-m$ even, otherwise	
(4) $K(M_2)$	(*a*) $\begin{pmatrix} \exp[(\lambda+i\theta)/2] & 0 \\ 0 & \exp[-(\lambda+i\theta)/2] \end{pmatrix}$	$m=2h$, $n=2f$ with $(h+f)/2$ odd	$z^h(\bar{z})^f$
	(*b*) $\begin{pmatrix} \exp[(\lambda+i\theta)/2] & 0 \\ 0 & \exp[-(\lambda+i\theta)/2] \end{pmatrix}$ and $\begin{pmatrix} 0 & \beta \\ -1/\beta & 0 \end{pmatrix}$	$m=2h$, $n=2f$ with $(h+f)/2$ even	$z^h(\bar{z})^f$

(5) $K(\dot{M}_1)$	$\begin{pmatrix}\alpha & \beta \\ -\bar{\beta} & \bar{\alpha}\end{pmatrix}$, $\lvert\alpha\rvert^2+\lvert\beta\rvert^2=1$	$m=n$	$(1+z\bar{z})^m$
(6) $K(\dot{E}_1)$	(*a*) $\begin{pmatrix}\alpha & \beta \\ \bar{\beta} & \bar{\alpha}\end{pmatrix}$, $\lvert\alpha\rvert^2-\lvert\beta\rvert^2=1$	$m=n$, odd	$(1-z\bar{z})^m$
	(*b*) $\begin{pmatrix}\alpha & \beta \\ \bar{\beta} & \bar{\alpha}\end{pmatrix}$ and $\begin{pmatrix}\beta & \alpha \\ -\bar{\alpha} & -\bar{\beta}\end{pmatrix}$, $\lvert\alpha\rvert^2-\lvert\beta\rvert^2=1$	$m=n$, even	$(1-z\bar{z})^m$
(7) $K(\dot{M}_2)$	$\begin{pmatrix}e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2}\end{pmatrix}$	$m-n$, even	†$\sum_f A_f z^{(m-n)/2+f}(\bar{z})^f$ $\max\{0,(n-m)/2\}\leqslant f\leqslant\min\{n,(m+n)/2\}$
(8) $K(\dot{E}_2)$	$\begin{pmatrix}\exp\left(\frac{\lambda}{2}+\frac{i\pi k}{2h-m}\right) & 0 \\ 0 & \exp\left(-\frac{\lambda}{2}-\frac{i\pi k}{2h-m}\right)\end{pmatrix}$	$m+n$, even	†all: $z^h(\bar{z})^{((m+n)/2-h)}$ with $\max\{0,(m-n)/2\}\leqslant h\leqslant\min\{m,(m+n)/2\}$
(9) Discrete	$\begin{pmatrix}\exp\left(i\frac{2\pi k}{2(h-f)-(m-n)}\right) & 0 \\ 0 & \exp\left(-i\frac{2\pi k}{2(h-f)-(m-n)}\right)\end{pmatrix}$	all m, n	†all $z^h(\bar{z})^f$
(10) $\mathrm{SL}(2, C)$	$\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$, $\alpha\delta-\beta\gamma=1$	all m, n	0

† Note that while all these vectors are fixed under the given group, hence also all linear combinations, the given group may not be the full little group of a given linear combination.

Theorem. (1) Every $T^{(m,n)}$ has a second-order invariant (a metric in the orbit space) which is symmetric only if $(m+n)$ is even, otherwise it is totally antisymmetric. (2) If $(m+n)$ is even $T^{(m,n)}$ has an additional third-order invariant. This is symmetric if $\frac{1}{2}(m+n)$ is also even. Otherwise it is totally antisymmetric.

Sketch of the proof. A tensor of degree d may be viewed as an element of the carrier space of the tensor product representation of $T^{(m,n)}$ with itself d times. An invariant tensor of degree d exists if and only if the tensor product representation contains the trivial representation $T^{(0,0)}$. By analysing the representations $T^{(m,n)}$ in terms of the representations of SU(2) we find that

$$T^{(j,k)} \subset T^{(m_1,n_1)} \otimes T^{(m_2,n_2)}, \qquad \text{for } j = |m_1 - m_2|, |m_1 - m_2| + 2, \ldots, m_1 + m_2,$$
$$k = |n_1 - n_2|, |n_1 - n_2| + 2, \ldots, n_1 + n_2,$$

and each such $T^{(j,k)}$ is contained exactly once. In order to study the symmetry properties of the tensors we derive the Clebsch–Gordan coefficients for tensor products of the $T^{(m,n)}$ representations. This is done by writing them, as before, in terms of representations of SU(2) they contain. In this way we reduce symmetry properties of SL(2, C) tensors to similar properties of SU(2) tensors (Weiss 1981).

We mentioned before that another approach to this is via the Lie algebra. By considering the commutation relations we find that the condition for an invariant tensor, that it is in the centre of the enveloping algebra, reduces to a system of linear equations given the degree d of the invariant tensor and the particular $T^{(m,n)}$. Using this we may write an algorithm for generating arbitrary invariant tensors. Unfortunately, the algorithm is limited practically in that the amount of computer memory required grows rapidly with the dimension of the space and the degree of the tensor. We do find, however, the general form of the metric. Under $T^{(m,n)}$ the contravariant tensor

$$u(z, \bar{z}) = \sum_{h=0}^{m} \sum_{f=0}^{n} \binom{m}{h}\binom{n}{f} (-1)^{h+f} z^h \bar{z}^f \otimes z^{m-h} \bar{z}^{n-f}$$

is invariant. These comments may suggest the difficulty and depth of the problem of the determination of the invariants. Yet we expect this study to yield much fruit both in that the tensors will shed light on the physics of the higher-dimensional spaces, much as the Minkowski metric divides space–time into space-like, light-like and time-like directions, and in that the invariant tensors themselves represent absolute physical quantities in these spaces such as the speed of light.

5. An example: $\mathrm{P}^{(2,2)}$

An examination of table 1 shows that only the groups $\mathrm{P}^{(m,n)}$ with $m = n$ have the little groups SO(3), SO(2, 1) and E_2. From this we might expect only these spaces to have time-like, null and space-like directions. The dimensions of these spaces are all squares $N = (m+1)^2$. We see that $\mathrm{P}^{(1,1)}$ with dimension $4 = 2^2$ has the smallest non-trivial dimension in this way. $\mathrm{P}^{(2,2)}$ with dimension nine is, therefore, the most direct generalisation of the Poincaré group. From the theorems above we know there are symmetric second- and third-order invariant tensors under $R^{(2,2)}$, hence there are invariant polynomials of degree two and three. Further, using the computer program mentioned above, we found two fourth-degree invariant polynomials independent of these. We also found that there is no independent fifth-degree invariant polynomial. There was not enough computer memory to investigate sixth-order polynomials.

We list the invariant polynomials in terms of their action on a vector $u = u_{hf}z^h(\bar{z})^f$:

$$p_2(u) = 8u_{22}u_{00} - 4u_{21}u_{01} + 8u_{20}u_{02} - 4u_{12}u_{10} + (u_{11})^2.$$

$$p_3(u) = u_{00}u_{11}u_{22} - u_{01}u_{10}u_{22} - u_{00}u_{12}u_{21} + u_{02}u_{10}u_{21} + u_{01}u_{12}u_{20} - u_{02}u_{11}u_{20},$$

$$\begin{aligned} p_{4.1}(u) = {} & 192u_{22}u_{20}u_{02}u_{00} - 32u_{22}u_{20}(u_{12})^2 - 32(u_{21})^2u_{02}u_{00} + 16(u_{21})^2(u_{01})^2 \\ & - 32u_{22}(u_{10})^2u_{02} + 16u_{22}u_{11}u_{10}u_{01} - 32u_{22}u_{12}u_{10}u_{00} + 16u_{21}u_{11}u_{10}u_{02} \\ & - 8u_{21}(u_{11})^2u_{01} + 16u_{21}u_{12}u_{11}u_{00} - 32u_{20}u_{12}u_{10}u_{02} + 16u_{20}u_{12}u_{11}u_{01} \\ & - 32u_{20}(u_{12})^2u_{00} - 32u_{22}u_{21}u_{01}u_{00} + 32(u_{22}u_{00})^2 - 32u_{21}u_{20}u_{02}u_{01} \\ & + 32(u_{20}u_{02})^2 + 16(u_{12}u_{10})^2 - 8u_{12}(u_{11})^2u_{10} + (u_{11})^4, \end{aligned}$$

$$\begin{aligned} P_{4.2}(u) = {} & 48u_{22}(u_{11})^2u_{00} - 32u_{22}u_{11}u_{10}u_{01} + 128u_{22}(u_{10})^2u_{02} - 32u_{21}u_{12}u_{11}u_{00} \\ & + 96u_{21}u_{12}u_{10}u_{01} - 32u_{21}(u_{11})^2u_{01} + 128u_{21}u_{11}u_{10}u_{02} + 64u_{20}(u_{12})^2u_{00} \\ & - 32u_{20}u_{12}u_{11}u_{01} + 48u_{20}(u_{11})^2u_{02} - 128u_{22}u_{12}u_{10}u_{00} - 128u_{20}u_{12}u_{10}u_{02} \\ & + 64u_{22}u_{20}(u_{01})^2 - 128u_{22}u_{21}u_{01}u_{00} + 128(u_{22}u_{00})^2 - 128u_{21}u_{20}u_{02}u_{01} \\ & + 64(u_{21})^2u_{02}u_{00} + 16(u_{21}u_{01})^2 + 128(u_{20}u_{02})^2 - 8u_{12}(u_{11})^2u_{10} \\ & + 16(u_{12}u_{10})^2 + (u_{11})^4. \end{aligned}$$

This special case $P^{(2,2)}$ illustrates some new little groups that arise from these generalised groups and suggests physical implications. Going through table 1 one finds all four little groups of the Poincaré group $P^{(1,1)}$ appearing and in addition four groups with new connected components (3), (4), (7) and (8); and the discrete group Z_2. Table 2 illustrates this.

These new little groups may be seen as describing more complex elementary systems. $P^{(2,2)}$ is the Lorentz group acting on a nine-dimensional vector space

Table 2. New connected little groups in the case $P^{(2,2)}$.

Little group H (see table 1)	Characterisation in Minkowski space of the connected component of H	Fixed vectors in $\mathscr{P}$
(3*b*)	fixes a two-dimensional null space	Linear combinations of the form: $A(z-\bar{z})^2 + Bz^2\bar{z}^2$.
(3*c*)	fixes a two-dimensional null space	Linear combinations of the form: $A(z-\bar{z})^2 + B(z-\bar{z})z\bar{z} + Cz^2\bar{z}^2$
(7)	fixes a two-dimensional Minkowski space	Linear combinations of the form: $A\cdot 1 + Bz\bar{z} + Cz^2\bar{z}^2$
(8)	fixes a two-dimensional Euclidian space	Linear combinations of the form: $Az^2 + Bz\bar{z} + C\bar{z}^2$
(4)	preserves a two-dimensional Minkowski space	Multiples of $z\bar{z}$

analogous to the Lorentz group acting on a four-dimensional space–time in the Poincaré group $P^{(1,1)}$. A little group in $P^{(1,1)}$ describes an elementary system or particle according to its fixed vector a particle with a time-like momentum vector is a massive particle, one with a null momentum vector must be a photon or neutrino, and a particle with a space-like momentum vector is a tachyon. If we take these nine dimensions seriously as an alternate dimensionality of reality to the four dimensions of space–time we must give these dimensions physical meaning. This should come from the action of the group on the space. The geometry of the orbits is difficult to see, but the algebraic properties are clear. The fixed vectors of E_2, SU(2) and SU(1, 1) should be null, time-like and space-like directions, respectively, though time and space may have somewhat different meanings here. For example, squared lengths of time-like and space-like vectors have opposite signs in Minkowski space. One may take from table 1 our chosen fixed vectors for SU(2)[(5)$K(\dot{M}_1)$] and SU(1, 1)[(6)$K(\dot{E}_1)$] and evaluate the four invariant polynomials mentioned above at the points in the nine-dimensional space defined by these vectors. Neither the metric nor either of the two fourth-order invariants separate these time-like and space-like vectors. They do differ by a sign when evaluated by the third-order invariant.

There are three little groups which are characterised as fixing a two-dimensional subspace of Minkowski space, one for each of the three types: Euclidean, Minkowski and null. Each of these is the little group for a three-parameter family of vectors! This phenomenon is not seen at all in the Poincaré group. In the Poincaré group, while isomorphic forms of SU(2) fix all time-like vectors, a given form of SU(2) fixes only multiples of a single vector.

Let us consider the three-parameter family of directions in $R^9 \subset P^{(2,2)}$ whose little group is characterised in $P^{(1,1)}$ as fixing a two-dimensional null space generated by a light-like and orthogonal space-like vector. Thus, a system with one of these directions as its momentum vector in the nine-dimensional space has the group characterised in Minkowski space as fixing both a null and space-like direction as the group of all motions not affecting its nine-dimensional momentum vector. So this direction is associatated with a system characterised by a light-like and a space-like direction. Thus we may associate such a direction for example with a plane polarised light wave, a wave moving in a specific light-like direction with a particular space-like polarisation. A choice of null and space-like directions in the Minkowski space yields a three-parameter family of vectors in the nine-dimensional space. This seems to describe three different kinds of plane waves. Similarly, the directions with little groups characterised in Minkowski space as fixing a two-dimensional Minkowski or null space may be associated with plane matter waves or plane tachyon waves, respectively.

The group (4*b*) which preserves the space generated by a space-like and time-like direction presents more of a mystery in that directions are only preserved and not fixed. A system whose momentum vector in the nine-dimensional space has (4*b*) as its little group would be characterised by a space-like and a time-like direction, but in such a way that translation in these directions are symmetries. Such a system might be an infinite string, so that boosts in that direction are symmetries, or the complementary two-dimensional Euclidean space existing only for an instant.

Of course other physical interpretations and applications are possible. The Lorentz group is a fundamental group in mathematical physics and its action on the space of its irreducible representations seems very rich. We expect this study to be relevant to a broad range of areas.

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